Project 3: Using Structure and Identities to Rewrite Expressions

Variables can be used to represent general structures

You may be used to seeing properties (patterns that we observe to be true with certain types of numbers or expressions) written out as equations using variables. This is a way in which variables are standing in for certain structures. So, for example, you may have seen the following property:

$$a(b+c) = ab + ac$$

What this equation means, is that any time we see something with the structure $a(b+c)$ we can replace it with the structure $ab + ac$ (but in order to do that we first have to identify what a, b and c are equal to in the particular expression we are working with. But in algebra, a, b and c can stand in for numbers, for other variables like x, or for other more complicated expressions like $-2xy - 5$. Let's look at some examples so that we can get practice identifying the specific structure $a(b+c)$. All of the following expressions have the structure $a(b+c)$ as long as we choose the right values for a, b and c:

- $2(x+4)$ $\quad a = 2, \;\; b = x, \;\; c = 4$
- $3(2x-5)$ First we have to rewrite addition as subtraction: $2(2x + -5)$. Then $a = 3, b = 2x, c = -5$
- $\sqrt{2}\left(\sqrt{2}+\sqrt{7}\right)$ $\quad a = \sqrt{2}, \;\; b = \sqrt{2}, \;\; c = \sqrt{7}$ (Notice that in this case we happen to have $a = b$, which is allowed, although it will not happen most of the time.)
- $3xy(2xy^2 + 3(xy)^3)$ $\quad a = 3xy, \quad b = 2xy^2, \quad c = 3(xy)^3$
- $-2x\left(3x^2 + (3x+4)\right)$ $\quad a = -2x, \quad b = 3x^2, \quad c = (3x+4)$
- $(2x-1)(3x+7)$ $\quad a = (2x-1), \quad b = 3x, \quad c = 7$

<u>**Now you try! Identify the structure:**</u>

For each of the following expressions, look for the structure $(a+b)c$. Notice that this is similar to the examples above, but this is a <u>different</u> structure than those examples. For each one, identify what a, b and c would have to be equal to in order for the expression to have the structure $(a+b)c$:

1. $(3x+2)x$

2. $\left(\sqrt{3}-\sqrt{5}\right)\sqrt{2}$

3. $(2x-2)2$

4. $(2x+1)(3x-4)$

5. $\left(3xy^2 \quad + 4(x^2y)^3\right)3x^2y$

6. $\left((3x-1)+2x^2\right)2x^2$

The same structure means that ALL the symbols are the SAME

We have to be careful when looking at structures to be sure that ALL the symbols are EXACTLY the same. If they aren't the same (or they don't at least **mean the same thing**—for example two different symbols that both mean multiplication), then the two expressions (or equations) don't have the same structure and likely do not have the same meaning. For example:

- $a(b+c)=ab+ac$ has the same structure as $c(a+b)=ca+cb$ We have just replaced a with c, b with a, and c with b, but **everything else stays exactly the same.**
- $(a+b)c$ has the same structure as $(y+x)z$ We have just replaced a with y, b with x, and c with z, but **everything else stays exactly the same.**
- $(a+b)(c+d)$ has the same structure as $(a+b)\cdot(c+d)$ We have just rewritten the multiplication using the multiplication sign, but **everything else stays exactly the same.**
- $a-(b+c)$ has the same structure as $4-((2x)+(z^2))$ We have just replaced a with 4, b with $2x$, and c with z^2, but **everything else stays exactly the same.**
- $a(b+c)=ab+ac$ does NOT have the same structure as $(a+b)c=ac+bc$ (the multiplication and addition aren't in the same places)
- $a+b=b+a$ does NOT have the same structure as $a-b=a+-b$ (addition has been changed to subtraction)
- $(a+b)c$ does NOT have the same structure as $(a+b)^c$ (multiplication has been changed to an exponent)
- $(a+b)(c+d)$ does NOT have the same structure as $(a+b)+(c+d)$ (multiplication has been changed to addition)
- $a-(b+c)$ does NOT have the same structure as $a-b+c$ (parentheses are missing)
- $a+(b\cdot a)$ does NOT have the same structure as $p+(q\cdot r)$ (the first and last variables are no longer the same)

For each of these examples, we can see that if we copy every single symbol from the first equation or expression, there is no way we can get the second equation or expression, even if we change the names of all the variables. For two expressions to have the same structure, **every single symbol has to be copied exactly as it is in the original expression or equation**. If we are substituting something in for a variable, that variable itself may change, but **nothing outside of the substitution should change at all.**

Now you try! For each of the following expressions or equations, CIRCLE any expressions or equations that have the SAME structure as the original equation, and CROSS OUT WITH AN X any expressions or equations that do NOT have the same structure as the original equation:

6. $(x+y)(y-z)$
   $(x+y)(y+z)$
   $(x+y)\cdot(y-z)$
   $(x+y)+(y-z)$
   $(m+b)(b-q)$
   $(x+y)y-z$
   $(a+b)(c-d)$
   $(x+y)-(y-z)$
   $(x+y)/(y-z)$
   $x+yy-z$

7. $(y+z)x$
   $(y+z)+x$
   $(y+z)-x$
   $(y-z)x$
   $y+zx$
   $\left((z^2b)+(-3a-b)\right)(3ab)$
   $(y+x)z$
   $(a+b)a$
   $(y+z)^x$
   $y(z+x)$

Identities

In algebra, equations that describe properties or patterns are often called *identities*. **Identities describe an expression can be replaced with an equal or equivalent expression that has a different form.**

So, for example, consider this identity: $a + b = b + a$. This means that for any two things that we are adding together (numbers, simple expressions, more complex expressions—**anything that the order of operations allows us to group together**), we can always switch the order and the result will still be equal to the original expression.

The actual letters a and b aren't important. We could also write this identity as $x + y = y + x$ or as $p + q = q + p$.

Example:

A) Let's rewrite $a + b = b + a$ using the letter p for a and the letter q for b:

$a = p, \quad b = q$

$a + b = b + a$

$\overbrace{(\quad)}^{a} + \overbrace{(\quad)}^{b} = \overbrace{(\quad)}^{b} + \overbrace{(\quad)}^{a}$

$\overbrace{(p)}^{a} + \overbrace{(q)}^{b} = \overbrace{(q)}^{b} + \overbrace{(p)}^{a}$

So we have $\boxed{p + q = q + p}$.

Notice that this has the same structure as the original identity $a + b = b + a$.

Using identities to rewrite expressions

Identities allow us to rewrite expressions by replacing an expression with a different equivalent expression (that may be simpler or more useful for graphing or solving something later).

Technically **we can never simply change or move something in an expression.**

Technically, **we can only rewrite expressions by replacing the expression with an equivalent one**.

Definition: Two expressions are *equivalent* if and only if they will always be equal, for ALL possible values of the variable(s). (If expressions contain multiple variables, this means for any possible combination of variable values.)

Examples:

For each of these expressions, we need to identify what is taking the place of the various variables in the identity. Just like in previous projects, what takes the place of the variables might not just be numbers, but could be more complex expressions that themselves contain variables or multiple terms.

We also have to be careful that when choosing groupings within an expression, we don't change the underlying structure. So, for example, we cannot add or remove parentheses from an expression if it changes the underlying meaning of the expression.

B) Use the identity $a + b = b + a$ to replace the original expression with an equivalent expression that has a different form:

$$2x + 3y$$

$a = 2x, \quad b = 3y$

$a + b = b + a$

$\overbrace{(\quad)}^{a} + \overbrace{(\quad)}^{b} = \overbrace{(\quad)}^{b} + \overbrace{(\quad)}^{a}$

$\overbrace{(2x)}^{a} + \overbrace{(3y)}^{b} = \overbrace{(3y)}^{b} + \overbrace{(2x)}^{a}$

So: $\boxed{\mathbf{2x + 3y = 3y + 2x}}$

C) Use the identity $\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$ (whenever $c \neq 0$) to replace the original expression with an equivalent expression that has a different form:

$$\frac{2x^2 + 4y}{2}$$

$a = 2x^2,\ \ b = 4y,\ \ c = 2$ (Since $2 \neq 0$, we know that $c \neq 0$)

$\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$

$$\frac{\overbrace{(\ )}^{a}+\overbrace{(\ )}^{b}}{\underbrace{(\ )}_{c}} = \frac{\overbrace{(\ )}^{a}}{\underbrace{(\ )}_{c}} + \frac{\overbrace{(\ )}^{b}}{\underbrace{(\ )}_{c}}$$

$$\frac{\overbrace{(2x^2)}^{a}+\overbrace{(4y)}^{b}}{\overbrace{(2)}^{c}} = \frac{\overbrace{(2x^2)}^{a}}{\overbrace{(2)}^{c}} + \frac{\overbrace{(4y)}^{b}}{\overbrace{(2)}^{c}}$$

So: $\boxed{\frac{2x^2+4y}{2} = \frac{2x^2}{2} + \frac{4y}{2}}$

D) Use the identity $x^n = \underbrace{x \cdot \cdots \cdot x}_{n-many\ times}$ (when n is a positive whole number) to replace the original expression with an equivalent expression that has a different form:

$$(3xy - 4)^3$$

Important point! You may notice in this problem that we have an x in the identity and that we also have an x in the expression. This can be confusing. **These two x's are NOT the same, because they are coming from two totally <u>different</u> contexts. In the identity, the x is just standing in for ANY expression; whereas in the expression, the x may have a particular value or may represent a particular thing that is varying (for example: time, or distance).** In this case, it might be easier to <u>first</u> rewrite the identity with a different letter instead of using x. For example, we could rewrite the identity using the letter a in place of x to avoid the confusion:

$x^n = \underbrace{x \cdot \cdots \cdot x}_{n-many\ times} \quad \to \quad a^n = \underbrace{a \cdot \cdots \cdot a}_{n-many\ times}$

$a = 3xy - 4,\quad n = 3$ (Since 3 is a positive whole number, we know that n is a positive whole number)

$$\overbrace{(\ )}^{a}{}^{\overbrace{(\ )}^{n}} = \underbrace{\overbrace{(\ )}^{a} \cdot \cdots \cdot \overbrace{(\ )}^{a}}_{\overbrace{(\ )}^{n}-many\ times}$$

$$\overbrace{(3xy-4)}^{a}{}^{\overbrace{(3)}^{n}} = \underbrace{\overbrace{(3xy-4)}^{a} \cdot \cdots \cdot \overbrace{(3xy-4)}^{a}}_{\overbrace{(3)}^{n}-many\ times}$$

$$\overbrace{(3xy-4)}^{a}{}^{\overbrace{(3)}^{n}} = \underbrace{\overbrace{(3xy-4)}^{a} \cdot \overbrace{(3xy-4)}^{a} \cdot \overbrace{(3xy-4)}^{a}}_{\overbrace{(3)}^{n}-many\ times}$$

So: $\boxed{\mathbf{(3xy-4)^3 = (3xy-4)\cdot(3xy-4)\cdot(3xy-4)}}$

Use the identity $x^0 = 1$ (whenever $x \neq 0$) to replace the original expression with an equivalent expression that has a different form: $(2xy^2 - 3y)^0$ $\quad (where\ 2xy^2 - 3y \neq 0)$

We could rewrite the identity $x^0 = 1$ (whenever $x \neq 0$) with a instead of x to avoid the confusion between the two x's (one in the identity and another in the expression):

$x^0 = 1$ (whenever $x \neq 0$) $\quad \to \quad a^0 = 1$ (whenever $a \neq 0$)

$a = 2xy^2 - 3y$

$a^0 = 1$

$\overbrace{(\ )}^{a}{}^{0} = 1 \quad (as\ long\ as\ a \neq 0)$

$\overbrace{(2xy^2 - 3y)}^{a}{}^{0} = 1 \quad (because\ we\ know\ that\ 2xy^2 - 3y \neq 0)$

So: $\boxed{\mathbf{(2xy^2 - 3y)^0 = 1}}$

Now you try! For each of the following problems, use the given identity to rewrite the given expression in a different equivalent form. **Use the examples above as a model, writing out all of the same steps.**

8. Use the identity $a - b = a + -b$ to replace the original expression with an equivalent expression that has a different form:

$$2x - 3y$$

9. Use the identity $\frac{a \cdot b}{c} = a \cdot \frac{b}{c}$ (whenever $c \neq 0$) to replace the original expression with an equivalent expression that has a different form:

$$\frac{(2x^2)(4y)}{2}$$

10. Use the identity $nx = \underbrace{x + \cdots + x}_{n\text{−many times}}$ (when n is a positive whole number) to replace the original expression with an equivalent expression that has a different form:

$$3(xy - 4)$$

11. Use the identity $a(b + c) = ab + ac$ to replace the original expression with an equivalent expression that has a different form:

$$(2x - 1)(3x^2 + 7)$$

Does it matter whether an expression has the form of the right or the left side of an identity?

You may have noticed that in all the examples so far, we have taken something that has the structure of the left side of the identity and replaced it with whatever is on the right side of the identity. But we can also do the reverse, because:

If two things are equal, we can always replace one with the other! It doesn't matter which side of an equation is written on the left or the right—the equals sign in the middle just tells us that both sides are equal.

Examples:

F) Use the identity $x^{-n} = \frac{1}{x^n}$ (whenever $x \neq 0$) to replace the original expression with an equivalent expression that has a different form:

$$\frac{1}{\left(3p - \sqrt{r}\right)^8} \qquad (3p - \sqrt{r} \neq 0)$$

This expression has the structure of the **right** side of the identity $x^{-n} = \frac{1}{x^n}$ rather than the left side.

If it is helpful, we could rewrite the identity to look like this: $\frac{1}{x^n} = x^{-n}$ (whenever $x \neq 0$), because this is just another way of saying exactly the same thing—that the two sides are equal.

We can now use this identity to rewrite our expression $\frac{1}{(3p-\sqrt{r})^8}$:

$x = 3p - \sqrt{r}, \;\; n = 8$

$\frac{1}{x^n} = x^{-n}$

$$\frac{1}{\overbrace{(\ \ )}^{x}{}^{\overbrace{(\ )}^{n}}} = \overbrace{(\ \ )}^{x}{}^{-\overbrace{(\ )}^{n}}$$

$$\frac{1}{\overbrace{(3p-\sqrt{r})}^{x}{}^{\overbrace{(8)}^{n}}} = \overbrace{\left(3p - \sqrt{r}\right)}^{x}{}^{-\overbrace{(8)}^{n}}$$ (We know that $x \neq 0$ because $3p - \sqrt{r} \neq 0$.)

So $\boxed{\boldsymbol{\frac{1}{(3p-\sqrt{r})^8} = \left(3p - \sqrt{r}\right)^{-8}}}$

G) Use the identity $a(b + c) = ab + ac$ to replace the original expression with an equivalent expression that has a different form:

$$2x(3x^3y^5) + 2x(3y - 4x)$$

This expression has the structure of the **right** side of the identity $a(b + c) = ab + ac$ rather than the left side.

If it is helpful, we could rewrite the identity to look like this: $ab + ac = a(b + c)$, because this is just another way of saying exactly the same thing—that the two sides are equal.

We can now use this identity to rewrite our expression $2x(3x^3y^5) + 2x(3y - 4x)$:

$a = 2x, \;\; b = 3x^3y^5, \;\; c = 3y - 4x$

$ab + ac = a(b + c)$

$$\overbrace{(\ \ )}^{a}\overbrace{(\ \ )}^{b} + \overbrace{(\ \ )}^{a}\overbrace{(\ \ )}^{c} = \overbrace{(\ \ )}^{a}\left(\overbrace{(\ \ )}^{b} + \overbrace{(\ \ )}^{c}\right)$$

$$\overbrace{(2x)}^{a}\overbrace{(3x^3y^5)}^{b} + \overbrace{(2x)}^{a}\overbrace{(3y - 4x)}^{c} = \overbrace{(2x)}^{a}\left(\overbrace{(3x^3y^5)}^{b} + \overbrace{(3y - 4x)}^{c}\right)$$

So $\boxed{\boldsymbol{(2x)(3x^3y^5) + (2x)(3y - 4x) = (2x)\left((3x^3y^5) + (3y - 4x)\right)}}$

Now you try! For each of the following problems, use the given identity to rewrite the given expression in a different equivalent form. **Use the examples above as a model, writing out all of the same steps.**

12. Use the identity $a(b+c) = ab + ac$ to replace the original expression with an equivalent expression that has a different form:

$$3xy(a^2b^3) + 3xy(a-1)$$

13. Use the identity $\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$ (whenever $c \neq 0$) to replace the original expression with an equivalent expression that has a different form:

$$\frac{9x + 3\sqrt{x^3}}{3}$$

14. Use the identity $a \cdot \frac{1}{b} = \frac{a}{b}$ (whenever $b \neq 0$) to replace the original expression with an equivalent expression that has a different form:

$$\frac{pqr - 1}{2pq^2r}$$

15. Use the identity $ac + bc = (a+b)c$ to replace the original expression with an equivalent expression that has a different form:

$$(x+7)(2xy-1)$$